Tannakian Categories attached to abelian Varieties

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Let k be an algebraically closed field k, where k is either the algebraic closure of a finite field or a field of characteristic zero. Let l be a prime different from the characteristic of k.

Notations. For a variety X over k let $D_c^b(X, \overline{\mathbb{Q}}_l)$ denote the triangulated category of complexes of etale $\overline{\mathbb{Q}}_l$-sheaves on X in the sense of [5]. For a complex $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$ let $D(K)$ denote its Verdier dual, and $\mathcal{H}^\nu(K)$ denote its etale cohomology $\overline{\mathbb{Q}}_l$-sheaves with respect to the standard t-structure. The abelian subcategory $Perv(X)$ of middle perverse sheaves is the full subcategory of all $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$, for which K and its Verdier dual $D(K)$ are contained in the full subcategory ${}^pD^{\leq 0}(X)$ of semi-perverse sheaves, where $L \in D_c^b(X, \overline{\mathbb{Q}}_l)$ is semi-perverse if and only if $dim(S_\nu) \leq \nu$ holds for all integers $\nu \in \mathbb{Z}$, where S_ν denotes the support of the cohomology sheaf $\mathcal{H}^{-\nu}(L)$ of L.

If k is the algebraic closure of a finite field κ, then a complex K of etale $\overline{\mathbb{Q}}_l$-Weil sheaves is mixed of weight $\leq w$, if all its cohomology sheaves $\mathcal{H}^\nu(K)$ are mixed etale $\overline{\mathbb{Q}}_l$-sheaves with upper weights $w(\mathcal{H}^\nu(K)) - \nu \leq w$ for all integers ν. It is called pure of weight w, if K and its Verdier dual $D(K)$ are mixed of weight $\leq w$. Concerning base fields of characteristic zero, we assume mixed sheaves to be sheaves of geometric origin in the sense of the last chapter of [1], so we still dispose over the notion of the weight filtration and purity and Gabber's decomposition theorem in this case. In this sense let $Perv_m(X)$ denote the abelian category of mixed perverse sheaves on X. The full subcategory $P(X)$ of $Perv_m(X)$ of pure perverse sheaves is a semisimple abelian category.

Abelian varieties. Let X be an abelian variety X of dimension g over an algebraically closed field k. The addition law of the abelian variety $a : X \times X \to X$ defines the convolution product $K * L \in D_c^b(X, \overline{\mathbb{Q}}_l)$ of two complexes K and L in $D_c^b(X, \overline{\mathbb{Q}}_l)$ by the direct image

$$K * L \ = \ Ra_*(K \boxtimes L) \ .$$

For the skyscraper sheaf δ_0 concentrated at the zero element 0 notice $K * \delta_0 = K$.

Translation-invariant sheaf complexes. More generally $K * \delta_x = T_{-x}^*(K)$, where x is a closed k-valued point in X, δ_x the skyscraper sheaf with support in $\{x\}$ and where $T_x(y) = y + x$ denotes the translation $T_x : X \to X$ by x. In fact $T_y^*(K * L) \cong T_y^*(K) * L \cong K * T_y^*(L)$ holds for all $y \in X(k)$. For $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$ let $Aut(K)$ be the abstract group of all closed k-valued points x of X, for which $T_x^*(K) \cong K$ holds. A complex K is called translation-invariant, provided $Aut(K) = X(k)$. If $f : X \to Y$ is a surjective homomorphism between abelian varieties, then the direct image $Rf_*(K)$ of a translation-invariant complex is translation-invariant. As a consequence of the formulas above, the convolution of an arbitrary $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$ with a translation-invariant complex on X is a translation-invariant complex. A translation-invariant perverse sheaf K on X is of the form $K = E[g]$, for an ordinary etale translation-invariant $\overline{\mathbb{Q}}_l$-sheaf E. For a translation-invariant complex $K \in D_c^b(X, \overline{\mathbb{Q}}_l)$ the irreducible constituents of the perverse cohomology sheaves ${}^pH^\nu(K)$ are translation-invariant.

Multipliers. The subcategory $T(X)$ of $Perv(X)$ of all perverse sheaves, whose irreducible perverse constituents are translation-invariant, is a Serre subcategory of the abelian category $Perv(X)$. Let denote $\overline{Perv}(X)$ its abelian quotient category and $\overline{P}(X)$ the image of $P(X)$, which is a full subcategory of semisimple objects. The full subcategory of $D_c^b(X, \overline{\mathbb{Q}}_l)$ of all K, for which ${}^pH^\nu(K) \in T(X)$, is a thick subcategory of the triangulated category $D_c^b(X, \overline{\mathbb{Q}}_l)$. Let

$$\overline{D}_c^b(X, \overline{\mathbb{Q}}_l)$$

be the corresponding triangulated quotient category, which contains $\overline{Perv}(X)$. Then the convolution product

$$* : \ \overline{D}_c^b(X, \overline{\mathbb{Q}}_l) \times \overline{D}_c^b(X, \overline{\mathbb{Q}}_l) \to \overline{D}_c^b(X, \overline{\mathbb{Q}}_l)$$

still is well defined, by reasons indicated above.

Definition. *A perverse sheaf* K *on* X *is called a multiplier, if the convolution induced by* K

$$*K : \overline{D_c^b}(X,\overline{\mathbb{Q}_l}) \to \overline{D_c^b}(X,\overline{\mathbb{Q}_l})$$

preserves the abelian subcategory $\overline{Perv}(X)$.

Obvious from this definition are the following properties of multipliers: If K and L are multipliers, so are the product $K * L$ and the direct sum $K \oplus L$. Direct summands of multipliers are multipliers. If K is a multiplier, then the Verdier dual $D(K)$ is a multiplier and also the dual

$$K^\vee = (-id_X)^*(D(K)) \ .$$

<u>Examples</u>: 1) Skyscraper sheaves are multipliers 2) If $i : C \hookrightarrow X$ is a projective curve, which generates the abelian variety X, and E is an etale $\overline{\mathbb{Q}}_l$-sheaf on C with finite monodromy, then the intersection cohomology sheaf attached to (C, E) is a multiplier. 3) If $: Y \hookrightarrow X$ is a smooth ample divisor, then the intersection cohomology sheaf of Y is a multiplier.

The proofs. 1) is obvious. For 2) we gave in [7] a proof by reduction mod p using the Cebotarev density theorem and counting of points. Concerning 3) the morphism $j : U = X \backslash Y \hookrightarrow X$ is affine for ample divisors Y. Hence $\lambda_U = Rj_!\overline{\mathbb{Q}}_l[g]$ and $\lambda_Y = i_*\overline{\mathbb{Q}}_{l,Y}[g-1]$ are perverse sheaves, which coincide in $\overline{Perv}(X)$. The morphism $\pi = a\circ(j\times id_X)$ is affine. Indeed $W = \pi^{-1}(V)$ is affine for affine subsets V of X, W being isomorphic under the isomorphism $(u,v) \mapsto (u, u+v)$ of X^2 to the affine product $U \times V$. By the affine vanishing theorem of Artin: For perverse sheaves $L \in Perv(X)$ we get $\lambda_U \boxtimes L \in Perv(X^2)$ and ${}^pH^\nu(R\pi_!(\lambda_U \boxtimes L)) = 0$ for all $\nu < 0$. The distinguished triangle $\big(Ra_*(\lambda_Y \boxtimes L), R\pi_!(\lambda_U \boxtimes L), Ra_*(\delta_X \boxtimes L)\big)$ for $\delta_X = \overline{\mathbb{Q}}_{l,X}[g]$ and the corresponding long exact perverse cohomology sequence gives isomorphisms ${}^pH^{\nu-1}(\delta_X * L) \cong {}^pH^\nu(\lambda_Y * L)$ for the integers $\nu < 0$. Since $Ra_*(\delta_X \boxtimes L) = \delta_X * L$ is a direct sum of translates of constant perverse sheaves δ_X, we conclude ${}^pH^\nu(\lambda_Y * L)$ for $\nu < 0$ to be zero in $\overline{Perv}(X)$. For smooth Y the intersection cohomology sheaf is $\lambda_Y = i_*\overline{\mathbb{Q}}_{l,Y}[g-1]$, and it is self dual. Hence by Verdier duality $i_*\overline{\mathbb{Q}}_{l,Y}[g-1] * L$ has image in $\overline{Perv}(X)$. Thus $i_*\overline{\mathbb{Q}}_{l,Y}[g-1]$ is a multiplier. $\square$

Let $M(X) \subseteq P(X)$ denote the full category of semisimple multipliers. Let $\overline{M}(X)$ denote its image in the quotient category $\overline{P}(X)$ of $P(X)$. Then, by the

definition of multipliers, the convolution product preserves $\overline{M}(X)$

$$* : \ \overline{M}(X) \times \overline{M}(X) \to \overline{M}(X) \ .$$

Theorem. *With respect to this convolution product the category $\overline{M}(X)$ is a semisimple super-Tannakian $\overline{\mathbb{Q}}_l$-linear tensor category, hence as a tensor category $\overline{M}(X)$ is equivalent to the category of representations $Rep(G, \varepsilon)$ of a projective limit*

$$G = G(X)$$

of supergroups.

Outline of proof. The convolution product obviously satisfies the usual commutativity and associativity constraints compatible with unit objects. See [7] 2.1. By [7], corollary 3 furthermore one has functorial isomorphisms

$$Hom_{\overline{M}(X)}(K, L) \ \cong \ \Gamma_{\{0\}}(X, \mathcal{H}^0(K * L^\vee)^*) \ ,$$

where $\mathcal{H}^0$ denotes the degree zero cohomology sheaf and $\Gamma_{\{0\}}(X, -)$ sections with support in the neutral element. Let $L = K$ be simple and nonzero. Then the left side becomes $End_{\overline{M}(X)}(K) \cong \overline{\mathbb{Q}_l}$. On the other hand $K * L^\vee$ is a direct sum of a perverse sheaf P and translates of translation-invariant perverse sheaves. Hence $\mathcal{H}^0(K * L^\vee)^\vee)$ is the direct sum of a skyscraper sheaf S and translation-invariant etale sheaves. Therefore $\Gamma_{\{0\}}(X, \mathcal{H}^0(K * L^\vee)^\vee) = \Gamma_{\{0\}}(X, S)$. By a comparison of both sides therefore $S = \delta_0$. Notice δ_0 is the unit element 1 of the convolution product. Using the formula above we not only get

$$Hom_{\overline{M}(X)}(K, L) \ \cong \ Hom_{\overline{M}(X)}(K * L^\vee, 1) \ ,$$

but also find a nontrivial morphism

$$ev_K : K * K^\vee \to 1 \ .$$

By semisimplicity δ_0 is a direct summand of the complex $K * K^\vee$. In particular the Künneth formula implies, that the etale cohomology groups do not all vanish identically

$$H^\bullet(X, K) \neq 0 \ .$$

Therefore the arguments of [7] 2.6 show, that the simple perverse sheaf K is dualizable. Hence $\overline{M}(X)$ is a rigid $\overline{\mathbb{Q}_l}$-linear tensor category. Let $\mathcal{T}$ be a finitely

$\otimes$-generated tensor subcategory with generator say A. To show $\mathcal{T}$ is super-Tannakian, by [4] it is enough to show for all n

$$lenght_{\mathcal{T}}(A^{*n}) \leq N^n ,$$

where N is a suitable constant. For any $B \in \overline{M}(X)$ let B, by abuse of notation, also denote the perverse semisimple representative in $Perv(X)$ without translation invariant summand. Put $h(B,t) = \sum_\nu dim_{\overline{\mathbb{Q}}_l}(H^\nu(X,B))t^\nu$. Then $lenght_{\mathcal{T}}(B) \leq h(B,1)$, since every summand of B is a multiplier and therefore has nonvanishing cohomology. For $B = A^{*n}$ the Künneth formula gives $h(B,1) = h(A,1)^n$. Therefore the estimate above holds for $N = h(A,1)$. This completes the outline for the proof of the theorem. $\square$

Principally polarized abelian varieties. Suppose Y is a divisor in X defining a principal polarization. Suppose the intersection cohomology sheaf δ_Y of Y is a multiplier. Then a suitable translate of Y is symmetric, and again a multiplier. So we may assume $Y = -Y$ is symmetric. Let $\overline{M}(X,Y)$ denote the super-Tannakian subcategory of $\overline{M}(X)$ generated by δ_Y. The corresponding super-group $G(X,Y)$ attached to $\overline{M}(X,Y)$ acts on the super-space $W = \omega(\delta_Y)$ defined by the underlying super-fiber functor ω of $\overline{M}(X)$. By assumption δ_Y is self dual in the sense, that there exists an isomorphism $\varphi : \delta_Y^\vee \cong \delta_Y$. Obviously $\varphi^\vee = \pm\varphi$. This defines a nondegenerate pairing on W, and the action of $G(X,Y)$ on W respects this pairing.

Curves. If X is the Jacobian of smooth projective curve C of genus g over k, X carries a natural principal polarization $Y = W_{g-1}$. If we replace this divisor by a symmetric translate, then Y is a multiplier. The corresponding group $G(X,Y)$ is the semisimple algebraic group $G = Sp(2g-2,\overline{\mathbb{Q}}_l)/\mu_{g-1}[2]$ or $G = Sl(2g-2,\overline{\mathbb{Q}}_l)/\mu_{g-1}$ depending on whether the curve C is hyperelliptic or not. The representation W of $G(X,Y)$ defined by δ_Y as above is the unique irreducible $\overline{\mathbb{Q}}_l$-representation of $G(X,Y)$ of highest weight, which occurs in the $(g-1)$-th exterior power of the $(2g-2)$-dimensional standard representation of G. See [7], section 7.6.

Conjecture. One could expect, that a principal polarized abelian variety (X,Y) of dimension g is isomorphic to a Jacobian variety $(Jac(C), W_{g-1})$ of a smooth projective curve C (up to translates of the divisor Y in X as explained above) if and only if Y is a multiplier with corresponding super-Tannakian group $G(X,Y)$ equal to one of the two groups

$$Sp(2g-2,\overline{\mathbb{Q}}_l)/\mu_{g-1}[2] \ \text{ or } \ Sl(2g-2,\overline{\mathbb{Q}}_l)/\mu_{g-1} \ .$$

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